On Inextensional Vibrations of Thin Shells

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In this paper the nonsymmetric, free, elastic vibrations of thin domes of revolution are studied. It is assumed that the frequency is low. The asymptotic approximations previously given by the writer are used to estimate the general solution to the shell vibration equations at low frequencies. Approximations for the low natural frequencies and modes are derived systematically under a variety of edge conditions. Low natural frequencies are found only when the edge conditions impose no forces tangent to the shell surface. When the edge is free (and only then) Rayleigh's inextensional frequencies are recovered. For certain other edge conditions, new natural frequencies are found that are above Rayleigh's frequencies but still low compared, e.g., with the lowest membrane frequency. The displacement modes associated with these new frequencies are mostly of inextensional type. The general results are applied to estimate these new frequencies for spherical domes.

Introduction

The inextensional vibrations of thin shells were first studied by Lord Rayleigh [1],[1] and since that time his procedure has often been used to estimate natural frequencies for various shell shapes. The frequencies obtained by this procedure are much lower (for a thin shell) than those predicted by any other method and are therefore of great practical interest. For example, a recent paper by Goodier and McIvor [2] shows the important part played by the inextensional modes in the transient response of an elastic cylindrical shell.

Despite the importance of Rayleigh's procedure, there is good reason for skepticism about its generality. For example, Love [4] has shown that the modes in general satisfy neither the motion equations nor (with a few exceptions) the edge conditions. Arnold and Warburton [5] observed that Rayleigh's procedure gave good agreement with experiments in some cases, but that the agreement was very sensitive to the edge conditions. In the same vein, Forsberg [3] has emphasized the care that is needed in choosing the boundary conditions for an approximate analysis.

The present investigation is an attempt to clarify this situation by studying how the inextensional modes may be derived from the general shell theory. We do not assume (as is usually done) that the mode is inextensional. Rather, we merely assume that the frequency is low (in a sense that will later be made more precise) and derive inextensional modes from the general shell theory. This change of procedure is important for two reasons. First, we find that inextensional modes can be derived only for certain edge conditions, and this sheds light on the questions raised in [3-5]. Second, the new procedure leaves the way open to find all low frequencies, whereas Rayleigh's procedure is limited to frequencies for which the modal bending energy greatly exceeds the modal stretching energy. For certain edge conditions, we shall find inextensional modes with frequencies different from those obtained by Rayleigh.

To demonstrate the procedure in a context general enough to be convincing but simple enough to avoid unessential manipulations, we consider a general dome of revolution executing small, nonsymmetric vibrations. We shall use the approximations obtained by the author [6] to write down an approximate general solution of the differential equation system when the frequency is low. This solution is substituted into the boundary conditions, the resulting frequency determinant is solved, and the ratios of the arbitrary constants are found. This entire process is carried through for four different edge conditions, starting with a free edge and proceeding at each stage to the "freest" of the remaining edge conditions. The frequency increases with each new edge condition until we exhaust all edge conditions for which low frequencies can be found.

For the two freest edge conditions this procedure gives complete estimates of the mode but only an order-of-magnitude estimate of the frequency. To find frequency estimates we use Rayleigh's principle for these cases. In the other two cases, explicit estimates are found for the inextensional frequencies.

[1] Numbers in brackets designate References at end of paper.

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Nomenclature

For most of the quantities defined below, we list the equations in which they are defined or first occur.

$A_1, A_2, \ldots, A_8$ = arbitrary constants in general solution, (14) and (16)
a, a^* = angle function in asymptotic approximations, (16)
B_1, B_2, B_5, B_7 = modified arbitrary constants in general solution
b_s, b_θ = meridional and circumferential rotations, (2)
$c(a), c(a^*)$ = abbreviations for cos (a), cos (a^*), (16)
D = $dw/d\sigma$, (12)
E = Young's modulus
E_K, E_S, E_B = kinetic, stretching, and bending energies, (13)
$f(\sigma)$ = $\cot \phi/r_\theta$, (10)
G_n, G_s, G_v = coefficient functions in asymptotic approximations, (16)
H = coefficient function in asymptotic approximation, (16)
h = shell thickness, assumed constant
K = E_K/Ω^2, (13)
$k_{ss}, k_{\theta\theta}, k_{s\theta}$ = dimensionless meridional, circumferential, and torsional curvature changes, (3)
L_5, L_Ω = combinations of direct stresses arising in certain modes, (34) and (35)
$M(\sigma)$ = $\csc \phi/r_\theta$, (10)
m = circumferential wave number
$m_{ss}, m_{\theta\theta}, m_{s\theta}$ = dimensionless meridional, circumferential, and twisting moments, (5)

(Continued on next page)

The general formulas are applied to a spherical dome, and numerical results are obtained for the previously unknown inextensional frequencies.

Fundamental Equations and Solutions

We shall adopt as our starting point the equations of thin-shell theory propounded by Sanders [7] and modified by the inclusion of translational (but not rotational) inertia. We may write the system in dimensionless form as in [6].

$$\gamma_{ss} = u' + wr_s^{-1}, \qquad \gamma_{\theta\theta} = uf + vM + wr_\theta^{-1} \tag{1}$$
$$\gamma_{s\theta} = (1/2)(v' - uM - vf)$$

$$b_s = -w' + ur_s^{-1}, \qquad b_\theta = vr_\theta^{-1} + wM \tag{2}$$

$$k_{ss} = b_s', \qquad k_{\theta\theta} = fb_s + Mb_\theta$$
$$k_{s\theta} = (1/2)\{b_\theta' - Mb_s - fb_\theta + (1/2)(r_\theta^{-1} - r_s^{-1})(v' + vf + uM)\} \tag{3}$$

$$n_{ss} = \gamma_{ss} + \nu\gamma_{\theta\theta}, \qquad n_{\theta\theta} = \gamma_{\theta\theta} + \nu\gamma_{ss} \tag{4}$$
$$n_{s\theta} = (1 - \nu)\gamma_{s\theta}$$

$$m_{ss} = k_{ss} + \nu k_{\theta\theta}, \qquad m_{\theta\theta} = k_{\theta\theta} + \nu k_{ss} \tag{5}$$
$$m_{s\theta} = (1 - \nu)k_{s\theta}$$

$$q_s = m_{ss}' + f(m_{ss} - m_{\theta\theta}) + Mm_{s\theta} \tag{6}$$
$$q_\theta = m_{s\theta}' + 2fm_{s\theta} - Mm_{\theta\theta}$$

The motion equations are

$$(1 - \nu^2)^{-1}\{n_{ss}' + f(n_{ss} - n_{\theta\theta}) + Mn_{s\theta}\} + \Omega^2 u + \epsilon^2\{q_s r_s^{-1} + (1/2)Mm_{s\theta}(r_s^{-1} - r_\theta^{-1})\} = 0 \tag{7}$$

$$(1 - \nu^2)^{-1}\{n_{s\theta}' + 2fn_{s\theta} - Mn_{\theta\theta}\} + \Omega^2 v + \epsilon^2\{q_\theta r_\theta^{-1} + (1/2)[(r_\theta^{-1} - r_s^{-1})M_{s\theta}]'\} = 0 \tag{8}$$

$$(1 - \nu^2)^{-1}\{n_{ss}r_s^{-1} + n_{\theta\theta}r_\theta^{-1}\} - \Omega^2 w - \epsilon^2\{q_s' + fq_s + Mq_\theta\} = 0 \tag{9}$$

where the various definitions are given in the Nomenclature, and

$$\Omega = \omega R(\rho/E)^{1/2}$$
$$\epsilon^2 = h^2/[12R^2(1 - \nu^2)] \ll 1 \tag{10}$$
$$f(\sigma) = r_\theta^{-1} \cot \phi, \qquad M(\sigma) = mr_\theta^{-1} \csc \phi$$

The boundary conditions at an edge have been given by Sanders [7] and consist of prescribing

$$n_{ss} \text{ or } u$$
$$N_{s\theta} \text{ or } v \tag{11}$$
$$Q_s \text{ or } w$$
$$m_{ss} \text{ or } D$$

where

$$N_{s\theta} = n_{s\theta} + \epsilon^2(1/2)(1 - \nu^2)(3r_\theta^{-1} - r_s^{-1})m_{s\theta}$$
$$Q_s = q_s + Mm_{s\theta} \tag{12}$$
$$D = w' = -b_s + ur_s^{-1}$$

The principle of conservation of energy for the vibrating shell states that

$$E_K - E_S - E_B = 0$$

where

$$E_K = \Omega^2\int(u^2 + v^2 + w^2)r_\theta \sin \phi d\sigma = \Omega^2 K$$
$$E_S = (1 - \nu^2)^{-1}\int(n_{ss}\gamma_{ss} + n_{\theta\theta}\gamma_{\theta\theta} + 2n_{s\theta}\gamma_{s\theta})r_\theta \sin \phi d\sigma \tag{13}$$
$$E_B = \epsilon^2\int(m_{ss}k_{ss} + m_{\theta\theta}k_{\theta\theta} + 2m_{s\theta}k_{s\theta})r_\theta \sin \phi d\sigma$$

and the integrals are extended over a meridian.

Nomenclature

$N_{s\theta}$ = modified dimensionless shear stress arising in boundary conditions, (11) and (12)

$n_{ss}, n_{\theta\theta}, n_{s\theta}$ = dimensionless meridional, circumferential, and shearing direct stresses, (4)

$n_{ss}^{(\epsilon)}, n_{\theta\theta}^{(\epsilon)}, n_{s\theta}^{(\epsilon)}$ = contribution of bending terms to direct stresses of inextensional solution

$n_{ss}^{(\Omega)}, n_{\theta\theta}^{(\Omega)}, n_{s\theta}^{(\Omega)}$ = contribution of inertia terms to direct stresses of inextensional solution

$n_{ss}^{(\lambda)}, n_{s\theta}^{(\lambda)}$ = $n_{ss}^{(\epsilon)}/r_\theta^2$, $n_{s\theta}^{(\epsilon)}/r_\theta^2$, respectively

Q_s = modified meridional shear force arising in boundary conditions, (11) and (12)

q_s, q_θ = dimensionless meridional and circumferential shear forces, (11) and (12)

R = characteristic radius of curvature, (10)

r_s, r_θ = dimensionless principal radii of curvature, (1)

$s(a), s(a^*)$ = abbreviations for $\sin a$, $\sin a^*$, (16)

u, v, w = dimensionless meridional, circumferential, and normal (outward) displacements, (1)

$X_{12}, X_{22}, X_{32}, X_{13}, X_{23}, X_{33}, X_{14}, X_{24}, X_{34}$ = elements of frequency determinant

$x(\sigma)$ = rapidly changing function arising in asymptotic approximations (15) and (17)

z = $n_{ss}^{(\Omega)} + n_{s\theta}^{(\Omega)}$, (45)

$\alpha_{\text{I}}, \alpha_{\text{II}}, \alpha_{\text{III}}, \alpha_{\text{IV}}$ = factors in expressions for inextensional frequencies, (27), (33), and (35)

β_5 = constant in determination of mode, (28)

γ = $x/2^{1/2}$, (16)

$\gamma_{ss}, \gamma_{\theta\theta}, \gamma_{s\theta}$ = meridional, circumferential, and shear strains, (1)

$\Delta(\sigma)$ = function arising in evaluation of E_s

ϵ = small dimensionless parameter, $(h/R)[12(1 - \nu^2)]^{-1/2}$

ζ = rapidly varying function, (31)

η_{11}, η_{21} = functions arising from higher-order terms in asymptotic approximations

θ = circumferential angle

$\Lambda(\sigma)$ = $dx/d\sigma$, (18)

λ = $\epsilon^{-1/2}$, large parameter, (15)

ν = Poisson ratio, (4) and (5)

ρ = mass density of shell material, (10)

σ = dimensionless meridional arc length

σ_0 = σ at shell edge

$\phi(\sigma)$ = angle between axis of revolution and shell normal

ϕ_0 = $\phi(\sigma_0)$

$\chi(\sigma)$ = functions occurring in modes, (30), (32), (34) and (36)

Ω = dimensionless circular frequency, (10)

ω = circular frequency

$(\;)'$ = $d(\;)/d\sigma$

The classical shell theory embodied in (1)–(9) neglects effects of transverse shear, thickness change, and rotational inertia and therefore can treat accurately only wavelengths much greater than the shell thickness, which implies

$$m\epsilon \ll 1.$$

This differential equation system is linear, of eighth order, and has singularities where $\sin\phi = 0$. We assume that $\sin\phi = 0$ at, and only at, the axis, and that the apex of the dome is of second degree.

The analysis depends on approximations to the eight solutions of this system which are described in [6] and will be summarized here. The essential step is an asymptotic analysis very similar to the one given earlier by the writer for the axisymmetric case [8] but now valid only when

$$m^2\epsilon \ll 1,$$

which we henceforth assume to be true.[2] It is found that four of the solutions usually vary rapidly with σ (i.e., along a meridian) and are called bending solutions, and four vary much more slowly and are called membrane or inextensional solutions. Two solutions of each type are singular at the dome apex, where $\sin\phi = 0$.

We shall now list the approximations to the four bending solutions, first near $\sin\phi = 0$, then for $\sin\phi \neq 0$. The latter are linear combinations of the approximations obtained in [6] for the case $\Omega r_\theta < 1$.

For $\sin\phi \simeq 0$:

$$w \simeq A_1 \operatorname{ber}_m(x) + A_2 \operatorname{bei}_m(x) + A_3 \ker_m(x) + A_4 \operatorname{kei}_m(x) \quad (14)$$

$$x = (1 - \Omega^2)^{1/4}\lambda\phi, \qquad \lambda = \epsilon^{-1/2} \gg 1 \quad (15)$$

For $\sin\phi \neq 0$:

$$\begin{bmatrix} w \\ n_{\theta\theta} \\ b_\theta \end{bmatrix} \simeq H \begin{bmatrix} 1 \\ G_n \\ M \end{bmatrix} \{A_1 e^\gamma c(a) + A_2 e^\gamma s(a) + A_3 \pi e^{-\gamma} c(a^*) - A_4 \pi e^{-\gamma} s(a^*)\}$$

$$\begin{bmatrix} u \\ n_{ss} \\ n_{s\theta} \\ q_s \end{bmatrix} \simeq \Lambda^{-1} H \begin{bmatrix} -G_u \\ fG_n \\ MG_n \\ \Lambda^4 \end{bmatrix} \{A_1 e^\gamma s(a^*) - A_2 e^\gamma c(a^*) + A_3 \pi e^{-\gamma} s(a) + A_4 \pi e^{-\gamma} c(a)\}$$

$$\begin{bmatrix} b_s \\ m_{s\theta} \end{bmatrix} \simeq \Lambda H \begin{bmatrix} -1 \\ (1-\nu)M \end{bmatrix} \{A_1 e^\gamma c(a^*) + A_2 e^\gamma s(a^*) - A_3 \pi e^{-\gamma} c(a) + A_4 \pi e^{-\gamma} s(a)\} \quad (16)$$

$$\begin{bmatrix} v \\ m_{ss} \\ m_{\theta\theta} \\ q_\theta \end{bmatrix} \simeq \Lambda^2 H \begin{bmatrix} -\Lambda^{-4} M G_v \\ -1 \\ -\nu \\ M \end{bmatrix} \{-A_1 e^\gamma s(a) + A_2 e^\gamma c(a) + A_3 \pi e^{-\gamma} s(a^*) + A_4 \pi e^{-\gamma} c(a^*)\}$$

where the A's are arbitrary constants and

$$x \equiv \lambda \int_{\sigma'=0}^{\sigma'=\sigma} (r_\theta^{-2} - \Omega^2)^{1/4} d\sigma' \quad (17)$$

$$\Lambda \equiv dx/d\sigma = \lambda(r_\theta^{-2} - \Omega^2)^{1/4} \gg 1 \quad (18)$$

$$H \equiv [(1 - \Omega^2)\epsilon]^{1/4} (2\pi r_\theta \sin\phi)^{-1/2} (r_\theta^{-2} - \Omega^2)^{-3/8}$$

$$\gamma \equiv 2^{-1/2}x, \qquad a \equiv \gamma - (\pi/8) + (1/2)m\pi, \qquad a^* \equiv a + (1/4)\pi$$

$$c(a) \equiv \cos a, \qquad s(a) \equiv \sin a$$

$$c(a^*) \equiv \cos a^* \qquad s(a^*) \equiv \sin a^*$$

$$G_u \equiv r_s^{-1} + \nu r_\theta^{-1}, \qquad G_n \equiv (1 - \nu^2) r_\theta^{-1}$$

$$G_v \equiv (2 + \nu) r_\theta^{-1} - r_s^{-1}$$

[2] Note that this is a more stringent inequality on m than the preceding one.

In deriving these formulas we have assumed that σ is measured from the apex of the dome. Also, R is chosen as the common value of the principal radii of curvature at the apex. Then $r_s(0) = r_\theta(0) = 1$ and the definition of x for $\sin\phi \simeq 0$ is a continuation of that for $\sin\phi \neq 0$.

In general, we shall neglect terms that are $0(\lambda^{-1})$ compared with terms that are $0(1)$. The asymptotic analysis shows that the errors in the approximations (14) and (16) are all $0(\lambda^{-1})$, and we shall neglect them henceforth. Moreover, if we expand these approximations in powers of Ω^2, we see that the static asymptotic approximations (the "edge-effect" approximations) are obtained when

$$\Omega^2 \leq 0(\lambda^{-2}).$$

The approximations for the remaining four solutions are also described in [6]. These are all membrane solutions when Ω^2 is not small but, as $\Omega^2 \to 0$, they must separate into two pairs of solutions, one pair (labeled 5 and 6) that approach the two static membrane solutions, and another pair (labeled 7 and 8) that approach the two approximately inextensional, static solutions. To be more precise, we now assume that the frequency is low, by which we mean

$$\Omega^2 \leq 0(\lambda^{-1}). \quad (19)$$

The solutions 5 and 6, the membrane solutions, are found in general by solving the system (1)–(9) with ϵ^2 set to zero in (7)–(9). When $\Omega^2 \leq 0(1)$, all the dimensionless variables associated with these two solutions are $0(1)$; i.e., the variables do not increase in size under differentiation. Hence, when $\Omega^2 \leq 0(\lambda^{-1})$, the inertia terms in (7)–(9) are negligible according to our criterion, and the membrane solutions are approximately just the static ones.

In estimating the inextensional solutions when $\Omega^2 \leq 0(\lambda^{-1})$ we are guided by the static case (see Love [4]). The displacements are obtained by solving (1) with $\gamma_{ss} = \gamma_{\theta\theta} = \gamma_{s\theta} = 0$, and all the other variables except n_{ss}, $n_{\theta\theta}$, $n_{s\theta}$ are then found from (2), (3), (5), and (6). These variables are all of the same order, say, $0(1)$, and are exactly the functions obtained in the static case. To estimate the small direct stresses we solve the motion equations, (7)–(9), as a system of three linear, nonhomogeneous equations for n_{ss}, $n_{\theta\theta}$, $n_{s\theta}$, taking the bending and inertia terms as already known. This gives the same results as in the static case except for additional terms arising from the inertia terms in (7)–(9), i.e., the two inextensional solutions for the direct stresses may be written

$$\begin{bmatrix} n_{ss} \\ n_{\theta\theta} \\ n_{s\theta} \end{bmatrix} \simeq A_7 \begin{bmatrix} \Omega^2 n_{ss7}^{(\Omega)} + \epsilon^2 n_{ss7}^{(\epsilon)} \\ \Omega^2 n_{\theta\theta 7}^{(\Omega)} + \epsilon^2 n_{\theta\theta 7}^{(\epsilon)} \\ \Omega^2 n_{s\theta 7}^{(\Omega)} + \epsilon^2 n_{s\theta 7}^{(\epsilon)} \end{bmatrix} + A_8 \begin{bmatrix} \Omega^2 n_{ss8}^{(\Omega)} + \epsilon^2 n_{ss8}^{(\epsilon)} \\ \Omega^2 n_{\theta\theta 8}^{(\Omega)} + \epsilon^2 n_{\theta\theta 8}^{(\epsilon)} \\ \Omega^2 n_{s\theta 8}^{(\Omega)} + \epsilon^2 n_{s\theta 8}^{(\epsilon)} \end{bmatrix} \quad (20)$$

where the functions multiplying ϵ^2 give the static bending contribution and the functions multiplying Ω^2 give the contribution of the inertia terms. The functions $n_{ss7}^{(\Omega)}$, $n_{ss7}^{(\epsilon)}$, $n_{ss8}^{(\Omega)} \ldots$ are all $0(1)$ in general. It will eventually be seen that these estimates of the inertia contributions to the direct stresses are very important to our development.

Of the eight solutions, four are singular at $\phi = 0$ and must be discarded for a dome. We may take two of these as the membrane solution numbered 6 and the inextensional solution numbered 8, and we see from (14) that the remaining two are the bending solutions numbered 3 and 4. Thus we must take

$$A_3 = A_4 = A_6 = A_8 = 0. \quad (21)$$

We may now without confusion set

$$n_{ss7}^{(\Omega)} \equiv n_{ss}^{(\Omega)}, \qquad n_{ss7}^{(\epsilon)} \equiv n_{ss}^{(\epsilon)}$$

and, similarly, for $n_{\theta\theta}$ and $n_{s\theta}$. The approximate general solution may finally be written

$$w \simeq He^{\gamma}\{A_1c(a) + A_2s(a)\} + A_5w^{(5)} + A_7w^{(7)}$$
$$u \simeq \Lambda^{-1}G_uHe^{\gamma}\{-A_1s(a^*) + A_2c(a^*)\} + A_5u^{(5)} + A_7u^{(7)}$$
$$v \simeq \Lambda^{-2}MG_vHe^{\gamma}\{A_1s(a) - A_2c(a)\} + A_5v^{(5)} + A_7v^{(7)}$$
$$b_s \simeq \Lambda He^{\gamma}\{-A_1c(a^*) - A_2s(a^*)\} + A_5b_s^{(5)} + A_7b_s^{(7)}$$
$$b_\theta \simeq MHe^{\gamma}\{A_1c(a) + A_2s(a)\} + A_5b_\theta^{(5)} + A_7b_\theta^{(7)}$$
$$m_{ss} \simeq \Lambda^2He^{\gamma}\{A_1s(a) - A_2c(a)\} + A_5m_{ss}^{(5)} + A_7m_{ss}^{(7)}$$
$$m_{\theta\theta} \simeq \Lambda^2\nu He^{\gamma}\{A_1s(a) - A_2c(a)\} + A_5m_{\theta\theta}^{(5)} + A_7m_{\theta\theta}^{(7)}$$
$$m_{s\theta} \simeq \Lambda M(1-\nu)He^{\gamma}\{A_1c(a^*) + A_2s(a^*)\} + A_5m_{s\theta}^{(5)} + A_7m_{s\theta}^{(7)} \quad (22)$$
$$q_s \simeq \Lambda^3He^{\gamma}\{A_1s(a^*) - A_2c(a^*)\} + A_5q_s^{(5)} + A_7q_s^{(7)}$$
$$q_\theta \simeq \Lambda^2MHe^{\gamma}\{-A_1s(a) + A_2c(a)\} + A_5q_\theta^{(5)} + A_7q_\theta^{(7)}$$
$$n_{ss} \simeq \Lambda^{-1}fG_nHe^{\gamma}\{A_1s(a^*) - A_2c(a^*)\} + A_5n_{ss}^{(5)} + A_7\{\Omega^2n_{ss}^{(\Omega)} + \lambda^{-4}n_{ss}^{(\epsilon)}\}$$
$$n_{\theta\theta} \simeq G_nHe^{\gamma}\{A_1c(a) + A_2s(a)\} + A_5n_{\theta\theta}^{(5)} + A_7\{\Omega^2n_{\theta\theta}^{(\Omega)} + \lambda^{-4}n_{\theta\theta}^{(\epsilon)}\}$$
$$n_{s\theta} \simeq \Lambda^{-1}MG_nHe^{\gamma}\{A_1s(a^*) - A_2c(a^*)\} + A_5n_{s\theta}^{(5)} + A_7\{\Omega^2n_{s\theta}^{(\Omega)} + \lambda^{-4}n_{s\theta}^{(\epsilon)}\}$$

These approximations are accurate when $\Omega^2 \leq 0(\lambda^{-1})$, $m \geq 2$ and $\sin\phi \neq 0$. When $\Omega^2 \leq 0(\lambda^{-2})$, all the quantities are approximately static (independent of Ω) except the direct stresses of the inextensional solution. These formulas form the basis for our analysis of low frequencies and modes.

Calculation of Low Frequencies and Modes

In this section we shall put the general solution (22) into various sets of edge conditions and calculate the natural modes and frequencies. The derivation will be carried out in detail for two cases but results will be given for the rest. We shall begin with the case of a completely free edge and then consider successively "tighter" sets of edge conditions until the frequency is increased above the range, $\Omega^2 \leq 0(\lambda^{-1})$, in which (22) is applicable.

We assume the edge is at $\sigma = \sigma_0$, and $\sin\phi(\sigma_0) \neq 0$. A new set of constants, B_j, $j = 1, 2, 5, 7$ is introduced, defined by

$$B_1 = A_1H(\sigma_0)e^{\gamma(\sigma_0)}, \quad B_2 = A_2H(\sigma_0)e^{\gamma(\sigma_0)}$$
$$B_5 = A_5, \quad B_7 = A_7,$$

and we also set

$$\lambda^{-4}n_{ss}^{(\epsilon)} = \Lambda^{-4}r_\theta^{-2}n_{ss}^{(\epsilon)} \equiv \Lambda^{-4}n_{ss}^{(\lambda)}$$
$$\lambda^{-4}n_{s\theta}^{(\epsilon)} = \Lambda^{-4}r_\theta^{-2}n_{s\theta}^{(\epsilon)} \equiv \Lambda^{-4}n_{s\theta}^{(\lambda)}$$

In deducing the natural frequency and evaluating the constants it is to be understood that all quantities are evaluated at $\sigma = \sigma_0$.

Case (I): Free Edge. $n_{ss} = N_{s\theta} = Q_s = m_{ss} = 0$ **at** $\sigma = \sigma_0$. The four conditions are (keeping the leading terms only)

$$n_{ss} = B_1\Lambda^{-1}fG_ns(a^*) - B_2\Lambda^{-1}fG_nc(a^*) + B_5n_{ss}^{(5)} + B_7\{\Omega^2n_{ss}^{(\Omega)} + \Lambda^{-4}n_{ss}^{(\lambda)}\} = 0 \quad (23)$$
$$N_{s\theta} = B_1\Lambda^{-1}MG_ns(a^*) - B_2\Lambda^{-1}MG_nc(a^*) + B_5n_{s\theta}^{(5)} + B_7\{\Omega^2n_{s\theta}^{(\Omega)} + \Lambda^{-4}[n_{s\theta} + gm_{s\theta}^{(7)}]\} = 0 \quad (24)$$
$$Q_s = B_1\Lambda^3s(a^*) - B_2\Lambda^3c(a^*) + B_5[q_s^{(5)} + Mm_{s\theta}^{(5)}] + B_7\{q_s^{(7)} + Mm_{s\theta}^{(7)}\} = 0 \quad (25)$$
$$m_{ss} = B_1\Lambda^2s(a) - B_2\Lambda^2c(a) + B_5m_{ss}^{(5)} + B_7m_{ss}^{(7)} = 0 \quad (26)$$

where

$$g = (1/2)r_\theta^{-2}(1-\nu^2)(3r_\theta^{-1} - r_s^{-1})$$

If we eliminate B_1 using the condition (23), we obtain a three-square system that can be written in matrix form

$$\begin{bmatrix} \eta_{11} & X_{12} & (\Omega^2X_{13} + \Lambda^{-4}X_{14}) \\ \eta_{21} & X_{22} & (\Omega^2X_{23} + \Lambda^{-4}X_{24}) \\ 2^{-1/2} & X_{32}\Lambda^{-1} & (\Omega^2\Lambda^{-1}X_{33} + \Lambda^{-4}X_{34}) \end{bmatrix} \begin{bmatrix} B_2\Lambda^{-2} \\ B_5 \\ B_7 \end{bmatrix} = [0]$$

where

$$X_{12} = n_{s\theta}^{(5)} - n_{ss}^{(5)}(M/f), \quad X_{22} = -n_{ss}^{(5)}/(fG_n), \quad X_{32} = -X_{22}s(a)$$
$$X_{13} = n_{s\theta}^{(\Omega)} - n_{ss}^{(\Omega)}(M/f), \quad X_{23} = -n_{ss}^{(\Omega)}/(fG_n), \quad X_{33} = -X_{23}s(a)$$
$$X_{14} = n_{s\theta}^{(\lambda)} + gm_{s\theta}^{(7)} - n_{ss}^{(7)}(M/f)$$
$$X_{24} = -\{n_{ss}^{(\lambda)}/fG_n\} + q_s^{(7)} + Mm_{s\theta}^{(7)}$$
$$X_{34} = -m_{ss}^{(7)}s(a^*)$$

When B_1 is eliminated, the leading terms in the coefficients of B_2 cancel in equations (24) and (25). The dominant terms in these coefficients then arise from later terms in the asymptotic expansions of n_{ss}, $n_{s\theta}$, and q_s for the bending solutions. These are not known explicitly, but we know their orders of magnitude and designate the unknown functions η_{11} and η_{21}, both of which are $0(1)$.

The frequency is found by annulling the determinant of this system, with the result

$$\Omega^2\Lambda^4 = \frac{X_{14}X_{22} - X_{12}X_{24} + 2^{1/2}(\eta_{21}X_{12} - \eta_{11}X_{22})X_{34}}{X_{12}X_{23} - X_{13}X_{22}} = \alpha_I^2 \quad (27)$$

The ratios of the coefficients are found to be

$$B_1/B_7 = \Lambda^{-2}2^{1/2}m_{ss}^{(7)}c(a^*)$$
$$B_2/B_7 = \Lambda^{-2}2^{1/2}m_{ss}^{(7)}s(a^*) \quad (28)$$
$$B_5/B_7 = \Lambda^{-4}\beta_5$$

where $\beta_5 = 0(1)$ is a constant.

The denominator in the frequency condition (27) is

$$X_{12}X_{23} - X_{13}X_{22} = n_{ss}^{(5)}n_{s\theta}^{(\Omega)} - n_{s\theta}^{(5)}n_{ss}^{(\Omega)}$$

and cannot vanish because the direct stresses associated with the membrane and inextensional solutions must be linearly independent. Thus, in the range $\Omega^2 \leq 0(\lambda^{-1})$, the frequency condition can be satisfied only when

$$\Omega \simeq \alpha_I\Lambda^{-2}$$

Since $\Lambda \simeq \lambda r_\theta^{-1/2}$ whenever $\Omega^2 \leq 0(\lambda^{-1})$, we find that there is only one natural frequency for each $m \geq 2$ in the range $\Omega^2 \leq 0(\epsilon^{1/2})$, and it is given by

$$\Omega \simeq \alpha_I\epsilon r_\theta(\sigma_0). \quad (29)$$

We cannot determine α_I and β_5 because we do not know η_{11} and η_{21}, which are found from the second terms in the asymptotic expansions of n_{ss}, $n_{s\theta}$, and q_s. Hence (29) is not of much practical value in calculating the frequency.

However, a first approximation to the mode is completely determined by the coefficients obtained in (28), even though we do not know β_5 precisely.

$$
\left.\begin{aligned}
&w \simeq w^{(7)}(\sigma), \quad v \simeq v^{(7)}(\sigma), \quad u \simeq u^{(7)}(\sigma) \\
&b_s \simeq b_s^{(7)}(\sigma), \quad b_\theta \simeq b_\theta^{(7)}(\sigma) \\
&\begin{bmatrix} m_{ss} \\ m_{\theta\theta} \end{bmatrix} \simeq \chi \begin{bmatrix} 1 \\ \nu \end{bmatrix} \sin\{\zeta - (\pi/4)\} + \begin{bmatrix} m_{ss}^{(7)}(\sigma) \\ m_{\theta\theta}^{(7)}(\sigma) \end{bmatrix} \\
&m_{s\theta} \simeq m_{s\theta}^{(7)}(\sigma) \\
&q_s \simeq \Lambda(\sigma)\,\chi \sin\zeta \\
&q_\theta \simeq -\chi M(\sigma) \sin\{\zeta - (\pi/4)\} + q_\theta^{(7)}(\sigma) \\
&\begin{bmatrix} n_{ss} \\ n_{s\theta} \end{bmatrix} \simeq \Lambda^{-3}(\sigma)\chi G_n(\sigma) \begin{bmatrix} f(\sigma) \\ M(\sigma) \end{bmatrix} \sin\zeta \\
&n_{\theta\theta} \simeq \Lambda^{-2}(\sigma)\chi G_n(\sigma) \cos\{\zeta - (\pi/4)\}
\end{aligned}\right\} \quad (30)
$$

$$
\begin{aligned}
\chi &= 2^{1/2}\,\frac{H(\sigma)}{H(\sigma_0)} \left\{\frac{\Lambda(\sigma)}{\Lambda(\sigma_0)}\right\}^2 e^{\zeta} m_{ss}^{(7)}(\sigma_0) \\
&= 2^{1/2} \left\{\frac{r_\theta(\sigma_0)}{r_\theta(\sigma)}\right\}^{3/4} \left\{\frac{\sin\phi(\sigma_0)}{\sin\phi(\sigma)}\right\}^{1/2} e^{\zeta} m_{ss}^{(7)}(\sigma_0) \qquad (31) \\
\zeta &= 2^{-1/2}(x - x_0) = -2^{-1/2}\lambda \int_\sigma^{\sigma_0} \{r_\theta(\sigma')\}^{-1/2} d\sigma'
\end{aligned}
$$

It is noteworthy that all the quantities occurring in these formulas for the mode are static and relatively easy to evaluate. The displacements and rotations are dominated by the inextensional solution, the membrane solution is entirely negligible, and the bending (edge-effect) solutions have a strong influence on the stresslike quantities, making possible the satisfaction of all boundary conditions.

Although this procedure has yielded only an order-of-magnitude estimate for the frequency, it has delivered an estimate of the mode that is both more general and more complete than any previously known.

Case (II): $n_{ss} = N_{s\theta} = Q_s = D = 0$ **at** $\sigma = \sigma_0$. Among the possible edge conditions this is the freest except for the free edge of Case (I). The analysis strongly resembles that of Case (I), and we shall merely record the results. Only one frequency is found in the range $\Omega \leq 0(\lambda^{-1})$ for each $m \geq 2$, namely,

$$
\Omega \simeq \alpha_{\mathrm{II}} \Lambda^{-3/2} \simeq \alpha_{\mathrm{II}} \{\epsilon r_\theta(\sigma_0)\}^{3/4}
$$

$\alpha_{\mathrm{II}} = 0(1)$ cannot be found explicitly because of cancellation of the leading terms, as was true of α_{I}. A complete first approximation for the mode is found

$$
\left.\begin{aligned}
&w \simeq w^{(7)}(\sigma), \quad u \simeq u^{(7)}(\sigma) \quad v \simeq v^{(7)}(\sigma) \\
&b_\theta \simeq b_\theta^{(7)}(\sigma) \\
&b_s \simeq \chi \cos\zeta + b_s^{(7)}(\sigma) \\
&\begin{bmatrix} m_{ss} \\ m_{\theta\theta} \end{bmatrix} \simeq -\Lambda(\sigma)\chi \begin{bmatrix} 1 \\ \nu \end{bmatrix} \sin\{\zeta - (\pi/4)\} \\
&m_{s\theta} \simeq -(1-\nu)M(\sigma)\chi \cos\zeta + m_{s\theta}^{(7)}(\sigma) \\
&q_s \simeq -\Lambda^2(\sigma)\chi \sin\zeta \\
&q_\theta \simeq \Lambda(\sigma)M(\sigma)\chi \sin\{\zeta - (\pi/4)\} \\
&\begin{bmatrix} n_{ss} \\ n_{s\theta} \end{bmatrix} \simeq -\Lambda^{-2}(\sigma)G_n(\sigma)\chi \begin{bmatrix} f \\ M \end{bmatrix} \sin\zeta \\
&n_{\theta\theta} \simeq -\Lambda^{-1}(\sigma)G_n(\sigma)\chi \cos\{\zeta - (\pi/4)\}
\end{aligned}\right\} \quad (32)
$$

where ζ is defined as in (31), and

$$
\chi = \frac{H(\sigma)}{H(\sigma_0)}\,\frac{\Lambda(\sigma)}{\Lambda(\sigma_0)}\,D^{(7)}(\sigma_0)e^{\zeta} = \frac{r_\theta^{1/4}(\sigma_0)}{r_\theta^{1/4}(\sigma)}\,\frac{\sin^{1/2}\phi(\sigma_0)}{\sin^{1/2}\phi(\sigma)}\,D^{(7)}(\sigma_0)e^{\zeta}
$$

The frequency is somewhat higher than in Case (I). The modal displacements are wholly inextensional, and the stresslike quantities are almost entirely derived from the bending solutions.

We see that Cases (I) and (II) are quite similar. In neither case can we calculate the frequency directly but, in both cases, we have very good knowledge of the mode. However, if we use Rayleigh's principle, we can translate accurate information about the mode into accurate information about the frequency. This is exactly what Rayleigh did for spherical domes and cylinders, and we shall derive general formulas for domes with the edge conditions of these two cases in the section, "Applications of Rayleigh's Principle."

Case (III): $n_{ss} = N_{s\theta} = w = m_{ss} = 0$ **at** $\sigma = \sigma_0$. This is the freest of the remaining boundary conditions. The analysis proceeds as in Case (I) except that (25) is replaced by

$$
w = B_1 c(a) + B_2 s(a) + B_5 w^{(5)} + B_7 w^{(7)} = 0.
$$

After eliminating B_1, the matrix equation of the system is

$$
\begin{bmatrix}
\eta_{11} & X_{12} & (\Omega^2 X_{13} + \Lambda^{-4} X_{14}) \\
2^{-1/2}\Lambda^2 & \Lambda X_{22} & (\Omega^2 \Lambda X_{23} + X_{24}) \\
2^{-1/2} & \Lambda^{-1} X_{32} & (\Omega^2 \Lambda^{-1} X_{33} + \Lambda^{-} X_{34})
\end{bmatrix}
\begin{bmatrix} \Lambda^{-2} B_2 \\ B_5 \\ B_7 \end{bmatrix} = [0]
$$

where

$$
X_{22} = -\frac{n_{ss}^{(5)} c(a)}{f G_n}, \qquad X_{23} = -\frac{n_{ss}^{(\Omega)} c(a)}{f G_n}, \qquad X_{24} = w^{(7)} s(a^*)
$$

$$
X_{32} = \frac{n_{ss}^{(5)} s(a)}{f G_n}, \qquad X_{33} = \frac{n_{ss}^{(\Omega)} s(a)}{f G_n}
$$

and the remaining X's are defined as in Case (I). We find for the frequency

$$
\Omega^2 \Lambda \simeq \frac{X_{12} X_{24}}{X_{12}(X_{33} - X_{23}) + X_{13}(X_{22} - X_{32})} = \alpha_{\mathrm{III}}^2
$$

and after some reduction

$$
\alpha_{\mathrm{III}}^2 = 2^{-1/2} \left[w^{(7)} G_n \left\{ \frac{f n_{s\theta}^{(5)} - M n_{ss}^{(5)}}{n_{ss}^{(\Omega)} n_{s\theta}^{(5)} - n_{s\theta}^{(\Omega)} n_{ss}^{(5)}} \right\} \right]_{\sigma = \sigma_0} \qquad (33)
$$

$$
\Omega \simeq \alpha_{\mathrm{III}} \Lambda^{-1/2} \simeq \alpha_{\mathrm{III}} \{\epsilon r_\theta(\sigma_0)\}^{1/4}
$$

The frequency is now higher by a factor of roughly $\epsilon^{-1/2}$ than in Case (II). The mode is

$$
\left.\begin{aligned}
&w \simeq w^{(7)}(\sigma) - \chi \cos\zeta, \quad u \simeq u^{(7)}(\sigma), \\
&\qquad v = v^{(7)}(\sigma) \\
&b_s \simeq \Lambda(\sigma)\chi \cos\{\zeta + (\pi/4)\}, \\
&\qquad b_\theta = b_\theta^{(7)}(\sigma) - M\chi \cos\zeta \\
&\begin{bmatrix} m_{ss} \\ m_{\theta\theta} \end{bmatrix} \simeq -\Lambda^2(\sigma)\chi \begin{bmatrix} 1 \\ \nu \end{bmatrix} \sin\zeta, \\
&\qquad m_{s\theta} \simeq -\Lambda(\sigma)M(1-\nu)\chi \cos\{\zeta + (\pi/4)\} \\
&q_s \simeq -\Lambda^3(\sigma)\chi \sin\{\zeta + (\pi/4)\}, \\
&\qquad q_\theta \simeq \Lambda^2(\sigma)M\chi \sin\zeta \\
&\begin{bmatrix} n_{ss} \\ n_{s\theta} \end{bmatrix} \simeq -\Lambda^{-1}(\sigma)G_n(\sigma)\chi \begin{bmatrix} f \\ M \end{bmatrix} \sin\{\zeta + (\pi/4)\} \\
&\qquad + \Lambda^{-1}(\sigma_0)w^{(7)}(\sigma_0)G_n(\sigma_0)2^{-1/2} \\
&\qquad \times \left\{ L_5 \begin{bmatrix} n_{ss}^{(5)}(\sigma) \\ n_{s\theta}^{(5)}(\sigma) \end{bmatrix} + L_\Omega \begin{bmatrix} n_{ss}^{(\Omega)}(\sigma) \\ n_{s\theta}^{(\Omega)}(\sigma) \end{bmatrix} \right\} \\
&n_{\theta\theta} \simeq -G_n\chi \cos\zeta
\end{aligned}\right\} \quad (34)
$$

where

$$
\chi = \{H(\sigma)/H(\sigma_0)\} e^{\zeta} w^{(7)}(\sigma_0)
$$

$$
L_5 = \left[\frac{M n_{ss}^{(\Omega)} - f n_{s\theta}^{(\Omega)}}{n_{ss}^{(\Omega)} n_{s\theta}^{(5)} - n_{s\theta}^{(\Omega)} n_{ss}^{(5)}} \right]_{\sigma = \sigma_0}
$$

$$
L_\Omega = \left[\frac{f n_{s\theta}^{(5)} - M n_{ss}^{(5)}}{n_{ss}^{(\Omega)} n_{s\theta}^{(5)} - n_{s\theta}^{(\Omega)} n_{ss}^{(5)}} \right]_{\sigma = \sigma_0}
$$

This mode differs from those in the two preceding cases in two important ways. First, the displacements are no longer completely inextensional, for the bending solutions make a contribution to w near the edge. Second, the effect of the membrane solution is not now completely negligible but is felt in the formulas for the direct stresses.

Case (IV): $n_{ss} = N_{s\theta} = w = D = 0$ **at** $\sigma = \sigma_0$. The analysis is the same in this case as in Case (III) with an obvious change in the last boundary condition. The natural frequency is found to be

$$\Omega \simeq \alpha_{IV}\Lambda^{-1/2}$$

$$\alpha_{IV}^2 = 2^{1/2}\left[w^{(7)}G_n\left\{\frac{fn_{s\theta}^{(5)} - Mn_{ss}^{(5)}}{n_{ss}^{(\Omega)}n_{s\theta}^{(5)} - n_{s\theta}^{(\Omega)}n_{ss}^{(5)}}\right\}\right]_{\sigma=\sigma_0} \quad (35)$$

and the mode is

$$\left.\begin{aligned}
&w \simeq w^{(7)}(\sigma) + \chi \sin\{\zeta - (\pi/4)\},\\
&\qquad u \simeq u^{(7)}(\sigma), \qquad v \simeq v^{(7)}(\sigma)\\
&b_s \simeq -\Lambda(\sigma)\chi \sin\zeta,\\
&\qquad b_\theta \simeq b_\theta^{(7)}(\sigma) + \chi \sin\{\zeta - (\pi/4)\}\\
&\begin{bmatrix} m_{ss} \\ m_{\theta\theta} \end{bmatrix} \simeq -\Lambda^2(\sigma)\chi\begin{bmatrix} 1 \\ \nu \end{bmatrix}\cos\{\zeta - (\pi/4)\},\\
&\qquad m_{s\theta} \simeq \Lambda(\sigma)M\chi(1-\nu)\sin\zeta\\
&q_s \simeq -\Lambda^3(\sigma)\,\chi\cos\zeta,\\
&\qquad q_\theta \simeq \Lambda^2(\sigma)M\chi\cos\{\zeta - (\pi/4)\}\\
&\begin{bmatrix} n_{ss} \\ n_{s\theta} \end{bmatrix} \simeq -\Lambda^{-1}(\sigma)G_n(\sigma)\begin{bmatrix} f \\ M \end{bmatrix}\chi\cos\zeta\\
&\qquad + \Lambda^{-1}(\sigma_0)w^{(7)}(\sigma_0)G_n(\sigma_0)2^{-1/2}\\
&\qquad \times\left\{L_5\begin{bmatrix} n_{ss}^{(5)}(\sigma) \\ n_{s\theta}^{(5)}(\sigma)\end{bmatrix} + L_\Omega\begin{bmatrix} n_{ss}^{(\Omega)} \\ n_{s\theta}^{(\Omega)}\end{bmatrix}\right\}\\
&n_{\theta\theta} \simeq G_n(\sigma)\chi\sin\{\zeta - (\pi/4)\}
\end{aligned}\right\} \quad (36)$$

where

$$\chi = 2^{1/2}\{H(\sigma)/H(\sigma_0)\}e^{\zeta}w^{(7)}(\sigma_0)$$

This frequency and mode are qualitatively much like those in Case (III). We see from (35) and (33) that the frequency estimate in the present case is larger than in Case (III) by a simple factor $2^{1/2}$.

The edge conditions considered in Cases (I)–(IV) all have $n_{ss} = N_{s\theta} = 0$ at $\sigma = \sigma_0$; i.e., the edges of the shell have been free to move in directions tangent to the middle surface. We have now exhausted all the cases with this property. If we work out similar analyses for Case (V): $n_{ss} = v = Q_s = m_{ss} = 0$ at $\sigma = \sigma_0$, and Case (VI): $u = N_{s\theta} = Q_s = m_{ss} = 0$ at $\sigma = \sigma_0$, we see that natural frequencies in the range $\Omega^2 \leq 0(\lambda^{-1})$ cannot occur; i.e., for these edge conditions all the natural frequencies obey

$$\Omega^2 \geq 0(1).$$

But all the remaining edge conditions are obtained from (V) or (VI) by "tightening" some of the conditions. Hence, in all the remaining cases, the natural frequencies are at least as high as in Cases (V) or (VI). We conclude that only for Cases (I)–(IV) can we find natural frequencies in the range

$$\Omega^2 \leq 0(\lambda^{-1}).$$

Now, it is easy to see from the motion equations that inextensional solutions, i.e., solutions having the property that

$$n_{ss}, \quad n_{\theta\theta}, \quad n_{s\theta} \leq 0(\lambda^{-1})$$

and all other quantities are $0(1)$, cannot occur when $\Omega^2 \geq 0(1)$. Hence, for a dome, inextensional modes and frequencies can occur only in Cases (I)–(IV); i.e., only when the edge is free to move tangentially.

Applications of Rayleigh's Principle

In this section we shall see how estimates of the inextensional frequencies can be obtained for the Cases (I) and (II) by using Rayleigh's principle.

Rayleigh's principle states that for any field of displacements satisfying the edge conditions on displacements,

$$\Omega^2 \leq \Omega_E^2 = (E_s + E_B)/K \quad (37)$$

where E_s, E_B, and K are to be calculated from the given field of displacements by means of (13). The accuracy of the estimated frequency, Ω_E, depends on (and is usually much better than) the accuracy of the assumed displacement field. We must emphasize (because it is occasionally overlooked) the effect of the edge conditions on the displacements. If these edge conditions are not satisfied by the chosen displacement field, Ω_E may differ wildly from Ω and need not even be the larger of the two.

In applying Rayleigh's principle to Cases (I) and (II), we shall take as the trial displacements the approximate modes given for these cases by our previous analysis. From (13), (4), and (5), we have

$$E_S = (1-\nu^2)^{-2}\int_{\sigma=0}^{\sigma_0}\{n_{ss}^2 + n_{\theta\theta}^2 - 2\nu n_{ss}n_{\theta\theta} + 2(1+\nu)n_{s\theta}^2\}r_\theta \sin\phi d\sigma \quad (38)$$

$$E_B = \lambda^{-4}(1-\nu^2)^{-1}\int_{\sigma=0}^{\sigma_0}\{m_{ss}^2 + m_{\theta\theta}^2 - 2\nu m_{ss}m_{\theta\theta} + 2(1+\nu)m_{s\theta}^2\}r_\theta \sin\phi d\sigma \quad (39)$$

$$K = \int_{\sigma=0}^{\sigma_0}(u^2 + v^2 + w^2)r_\theta \sin\phi d\sigma \quad (40)$$

Referring to the formulas (30) and (32), for the modes in the two cases, we see that two kinds of terms occur; namely, terms of inextensional and edge-effect types. The integrals of the inextensional terms are of the same order as the terms themselves and cannot be evaluated explicitly until the shell shape is specified. The integrals of the edge-effect terms are smaller by an order of magnitude than the terms themselves and can be evaluated explicitly (though approximately) by use of the Laplace method for asymptotic approximation of definite integrals.

For example, in Case (II), (38) and (32) lead to

$$E_s \simeq (1-\nu^2)^{-2}\int_0^{\sigma_0} n_{\theta\theta}^2 r_\theta \sin\phi d\sigma$$

$$\simeq (1-\nu^2)^{-2}\int_0^{\sigma_0}[\Lambda^{-2}(\sigma)\chi^2(\sigma)r_\theta(\sigma)\sin\phi(\sigma)G_n^2(\sigma)] \times \cos^2[\zeta - (\pi/4)]d\sigma$$

$$\simeq \int_0^{\sigma_0}\{\Lambda^{-2}(\sigma)r_\theta^{-1}(\sigma)\sin\phi(\sigma)\Delta^2(\sigma)[D^{(7)}(\sigma)]^2\} \times e^{2\zeta}\cos^2[\zeta - (\pi/4)]d\sigma$$

where

$$\Delta(\sigma) = H(\sigma)\Lambda(\sigma)/[H(\sigma_0)\Lambda(\sigma_0)]$$

The function $e^{2\zeta}$ has the value unity for $\sigma = \sigma_0$ and decreases rapidly to zero as σ decreases from σ_0. Hence this integral is of Laplace type; i.e., only the region near $\sigma = \sigma_0$ contributes appreciably to the integral. We may therefore approximate it by

$$E_s \simeq \{\Lambda^{-2}r_\theta^{-1}[D^{(7)}]^2\sin\phi\}_{\sigma_0}(1/2)\int_0^{\sigma_0}e^{2\zeta}(1 + \sin 2\zeta)d\sigma$$

The integral in this expression can be evaluated approximately with the aid of (31) to give

$$E_s \simeq \{\Lambda^{-3} r_\theta^{-1}[D^{(7)}]^2 \sin\phi\}|_{\sigma_0}(2^{1/2}/8)$$

We know that $\Omega^2 = 0(\lambda^{-3})$, hence

$$\Lambda(\sigma_0) \simeq \lambda r_\theta^{-1/2}(\sigma_0).$$

Thus, finally, we find

$$E_s \simeq (1/8)\lambda^{-3}[2r_\theta(\sigma_0)]^{1/2}[D^{(7)}(\sigma_0)]^2 \sin\phi\ (\sigma_0)$$

In a similar manner E_B may be estimated,

$$E_B \simeq (3/8)\lambda^{-3}[2r_\theta(\sigma_0)]^{1/2}[D^{(7)}(\sigma_0)]^2 \sin\phi\ (\sigma_0)$$

Hence for Case (II)

$$\Omega_E^2 \simeq \frac{\lambda^{-3}[r_\theta(\sigma_0)/2]^{1/2}[D^{(7)}(\sigma_0)]^2 \sin\phi\ (\sigma_0)}{K^{(7)}} \tag{41}$$

where

$$K^{(7)} = \int_{\sigma=0}^{\sigma_0} \{[u^{(7)}(\sigma)]^2 + [v^{(7)}(\sigma)]^2 + [w^{(7)}(\sigma)]^2\} r_\theta \sin\phi d\sigma$$

Applying the same analysis in Case (I), we find

$$E_s = 0(\lambda^{-5})$$

$$E_B = E_B^{(7)} + 0\ (\lambda^{-5})$$

$$E_B^{(7)} = \lambda^{-4}\int_{\sigma=0}^{\sigma_0} \{[m_{ss}^{(7)}]^2 + [m_{\theta\theta}^{(7)}]^2 - 2\nu m_{ss}^{(7)} m_{\theta\theta}^{(7)} + 2(1+\nu)[m_{s\theta}^{(7)}]^2\} r_\theta \sin\phi d\sigma$$

$$K = K^{(7)}$$

Hence we obtain the estimate

$$\Omega_E^2 \simeq E_B^{(7)}/K^{(7)} = 0(\lambda^{-4}) \tag{42}$$

We see that in this case the estimate given by Rayleigh's principle can be derived solely from the inextensional displacements. This is not true of the estimate just obtained in Case (II), nor is it true of the Rayleigh estimates that are obtained for the inextensional frequencies in Cases (III) and (IV). Equation (42) is of course just the estimate that Rayleigh used to find the inextensional frequencies for a spherical dome. However, neither Rayleigh nor any subsequent investigator seems to have been sure of the conditions under which the estimate is accurate. We now see that it is accurate only when the edge of the dome is free.

Inextensional Frequencies for a Spherical Dome

In this section we carry out the calculation of the two lowest inextensional frequencies for a spherical dome under the edge conditions of Cases (II) and (III), using formulas (41) and (33), respectively.

For a spherical dome the inextensional solution that is finite at $\phi = 0$ has

$$\begin{aligned} u^{(7)} &= v^{(7)} = \sin\phi \tan^m(\phi/2), \qquad m \geq 2 \\ w^{(7)} &= -(m + \cos\phi)\tan^m(\phi/2) \\ D^{(7)} &= \{\sin\phi - m(m+\cos\phi)\csc\phi\}\tan^m(\phi/2) \end{aligned} \tag{43}$$

and the kinetic energy is given by (recall that ϕ_0 is the edge angle)

$$K^{(7)} = K(\phi_0, m) = \int_0^{\phi_0} \tan^{2m}(\phi/2)\{2\sin^2\phi + (m+\cos\phi)^2\}\sin\phi d\phi.$$

In Case (II) equation (41) reduces to

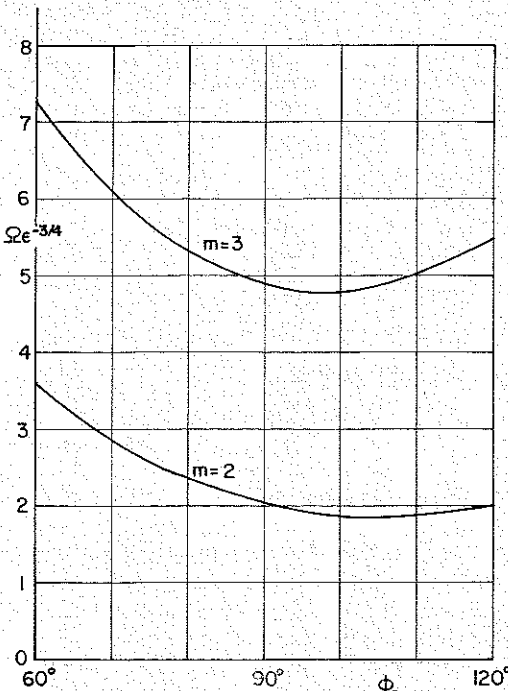

Fig. 1 Inextensional frequencies as functions of edge angle, ϕ_0, for the edge condition $n_{ss} = N_{s\theta} = Q_s = D = 0$ from equation (44)

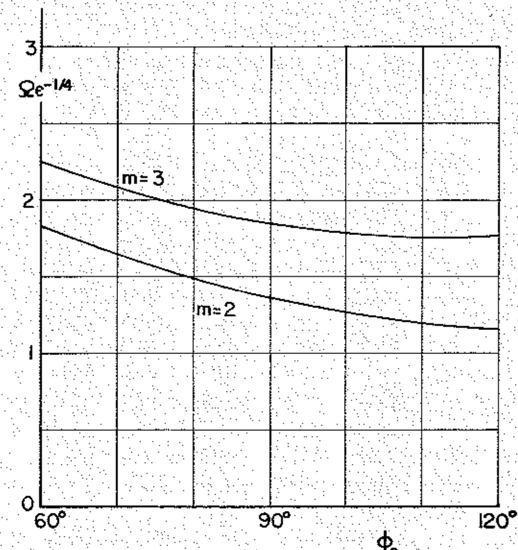

Fig. 2 Inextensional frequencies as functions of edge angle, ϕ_0, for the edge condition $n_{ss} = N_{s\theta} = w = m_{ss} = 0$ from equation (46)

$$\Omega_E^2 = \frac{2^{-1/2}\epsilon^{3/2}}{K(\phi_0, m)} \sin\phi_0 \tan^{2m}(\phi_0/2)\{\sin\phi_0 - m(m+\cos\phi_0)\csc\phi_0\}^2 \tag{44}$$

Fig. 1 shows graphs of the relations between Ω_E and ϕ_0 for $m = 2$ and 3, obtained from (44).

In Case (III) the frequency is given by (33). To evaluate this for a sphere, we observe first that the membrane solution finite at $\phi = 0$ has (see Love [4])

$$n_{s\theta}^{(5)} = -n_{ss}^{(5)},$$

and (33) reduces to

$$\alpha_{\mathrm{III}}^2 = \frac{-2^{-1/2}(1-\nu^2)(m+\cos\phi_0)^2 \csc\phi_0 \tan^m(\phi_0/2)}{\{n_{ss}^{(\Omega)} + n_{s\theta}^{(\Omega)}\}|_{\sigma_0}} \tag{45}$$

To find

$$z = n_{ss}^{(\Omega)} + n_{s\theta}^{(\Omega)}$$

we must solve the system of equations obtained from the motion equations, (7)–(9), by setting $\epsilon = 0$ and taking for u, v, and w the inextensional displacements (43). The governing equation is

$$\frac{dz}{d\phi} + (m + 2\cos\phi)\csc\phi\, z = -(1-\nu^2)\{u^{(7)} + v^{(7)} - w^{(7)}(m + \cos\phi)\csc\phi\}$$

and a particular solution (which is all we need) is

$$z(\phi_0) = -(1-\nu^2)K(\phi_0, m)\sin^{-2}\phi_0 \tan^{-m}(\phi_0/2)$$

Combining this with (45) and (33), we find

$$\Omega^2 \simeq \frac{\epsilon^{1/2}(m + \cos\phi_0)^2 \sin\phi_0 \tan^{2m}(\phi_0/2)}{2^{1/2}K(\phi_0, m)} \quad (46)$$

The frequencies predicted by (46) when $m = 2$ and 3 are shown in Fig. 2

The inextensional frequencies in Case (IV) are $2^{1/2}$ times those of Case (III).

Discussion

We may make the following comments about the results of the preceding sections.

(i) Although it has been customary to assume that inextensional modes of vibration could, in general, be derived from a classical shell theory (see Kalnins [9], for example), it seems not to have been done before. Our derivation is of course only approximate, but the results are reassuring.

(ii) The observation that inextensional modes can be found for a dome only when the edge is free to move tangentially is similar to a conclusion reached by Arnold and Warburton [5] for a cylinder.

(iii) The analysis shows that Rayleigh's procedure (i.e., using the static inextensional solution as trial displacements in Rayleigh's variational principle) works for a dome only when the edge is free. For cases (II)–(IV) the inextensional displacements do not satisfy the edge conditions required by the variational principle. Alternatively, using the inextensional displacements alone can work only when the modal energy of bending greatly exceeds that of stretching, a condition which the true modes do not satisfy in cases (II)–(IV).

(iv) Recently, Hwang [10] described difficulties in calculating the inextensional frequencies from the general solution for a free-edged hemisphere. The difficulty took the form of severe accuracy loss in computing the frequency determinant. The writer has commented elsewhere on this difficulty [11], but here we add that it may be partly related to the cancellation of the leading bending terms for n_{ss} and $n_{s\theta}$ in Cases (I) and (II). A sufficiently accurate solution would be free of this difficulty and, evidently, the calculations described by Kalnins [9] were successful in this case.

(v) The inextensional frequencies are acutely sensitive to the edge conditions. For example, if $h/R = 1/30$, the lowest inextensional frequency in Case (IV) may be larger than in Case (I) by a factor like 20–40. Conceivably this sensitivity could be useful for experimentally determining what the edge conditions actually are.

(vi) We have not considered boundary conditions of elastic constraint at the edge. In general, we may expect that these will produce frequencies lying between those associated with the two "pure" edge conditions that are combined to give the elastic condition. For example, the lowest natural frequency associated with the boundary condition

$$n_{ss} = N_{s\theta} = w = \xi m_{ss} + (1-\xi)D = 0,$$

where $0 \leq \xi \leq 1$, should satisfy

$$\alpha_{\mathrm{III}}\Lambda^{-1/2} \leq \Omega \leq \alpha_{\mathrm{IV}}\Lambda^{-1/2}.$$

Although we have chosen to demonstrate this procedure for domes, it ought to work equally well for shells with two edges. However, it remains always subject to the condition that $m^2\epsilon \ll 1$.

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